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LETTER TO THE EDITOR

Cluster structure of the Eden model and directed polymers in a random potential

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Abstract. The cluster structures of the Eden model and directed polymers in a random potential are described. The heights h_s and widths w_s of each cluster grows as $h_s \sim s^{\nu_\parallel}$ and $w_s \sim s^{\nu_\perp}$ with $\nu_\parallel = z/(d-1+z)$ and $\nu_\perp = 1/(d-1+z)$ in d dimensions where s is the size of the cluster and z is the dynamic critical exponent. Generalization to include the cluster size distribution is discussed.

Over recent years, there have been considerable efforts in studying the surface structure of the Eden model [1]. It is related through various mappings to other physical problems such as ballistic aggregation [2], the directed polymer in a random potential [3] and the Kardar–Parisi–Zhang equation [4]. Most efforts concentrated on studying the surface structure, especially on determining the exponents governing the surface fluctuations. In a finite system of size L, the standard deviation $\sigma(h)$ of the surface height starting from a flat substrate scales as [5]

$$\sigma(h) \sim L^{\chi} f(h/L^{z}) \tag{1}$$

where h is the average height of the surface and the scaling function $f(x)$ is $x^{\chi/z}$ for $x \ll 1$ and is constant for $x \gg 1$. Note that there is a correlation length $\xi \sim h^{1/z}$. The fluctuation of the surface height σ increases with h and becomes saturated when the correlation length ξ is of order L. In dimension $d = 2$ (we shall usually write $d = 2$ as $d = 1 + 1$ to indicate that there is one substrate direction and one growing direction) the exponents are known [4, 6] to be $\chi = 1/2$ and $z = 3/2$. However the values of the exponents in higher dimensions have not been settled yet.

Recently, Meakin [7] investigated the bulk structure by dividing the whole pattern of Eden growth into individual connected clusters. In off-lattice ballistic deposition, the incoming particle makes contact with only one particle in the previously deposited nucleation surface. So, the bulk structure can be divided into clusters (or trees) which are collections of particles connected to the same nucleation site. In a similar way, the bulk structure of the Eden model or the lattice version of ballistic deposition growth can be divided into clusters. For each cluster, the maximum width w_s and maximum height h_s follows scaling

$$w_s \sim s^{\nu_\perp} \qquad \text{and} \qquad h_s \sim s^{\nu_\parallel} \tag{2}$$

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where s is the size of the cluster. Alternatively, h_s can be defined as the standard deviation of the height from the substrate. The size distribution $n(s)$ which is the number of clusters of size s per unit area can be described by a power law [8]

$$n(s) \sim s^{-\tau} f(s^{2-\tau}/h) \tag{3}$$

where h is the average height. The exponents τ and $\nu_\parallel$ satisfy a scaling relation

$$\tau = 2 - \nu_\parallel \tag{4}$$

when the fractal dimensionality of the bulk is equal to the Euclidean dimensionality [8]. Here we report a relation between the exponents $\nu_\parallel$ and $\nu_\perp$ governing each cluster and the exponent z controlling the surface fluctuations.

We have already mentioned that there is a parallel (to the substrate) correlation length $\xi \sim h^{1/z}$ for $h \ll L^z$ in equation (1). Since the bulk structure in the off-lattice ballistic deposition model is divided into unconnected clusters, the height has a correlation up to the cluster width. So, we may regard the correlation length ξ as the width of the cluster giving a relation

$$w_s \sim \xi \sim h_s^{1/z} \tag{5}$$

for $h_s \ll L^z$. The correlation length ξ becomes proportional to the size of the system when the width of the cluster becomes of order L. From the definition of $\nu_\perp$ and $\nu_\parallel$ in equation (2), equation (5) implies

$$z\nu_\perp = \nu_\parallel. \tag{6}$$

If the individual clusters are compact, then the cluster size s is given by $s \sim w_s^{d-1} h_s \sim h_s^{(d-1+z)/z} \sim s^{\nu_\parallel (d-1+z)/z}$. With equations (4) and (6), we obtain

$$\nu_\parallel = \frac{z}{d-1+z} \qquad \nu_\perp = \frac{1}{d-1+z} \qquad \text{and} \qquad \tau = \frac{2d-2+z}{d-1+z} \tag{7}$$

for compact clusters where there is a relation [7]

$$\nu_\parallel + (d-1)\nu_\perp = 1. \tag{8}$$

Numerical results [7] $\nu_\parallel = 0.60 - 0.61$ and $\nu_\perp = 0.40 - 0.41$ for the ballistic model and the Eden model in d=1+1 are in an excellent agreement with 0.6 and 0.4 given by equation (7) with exact value $z = 3/2$. If we assume $z = 2(d+1)/(d+2)$ [9], then $\nu_\parallel = 4/9 \approx 0.444$ and $\nu_\perp = 5/18 \approx 0.278$ in $d = 2+1$. These are in a good agreement with the numerical results $\nu_\parallel = 0.45 - 0.47$ and $\nu_\perp = 0.27 - 0.29$ obtained from the growth models.

To check the power law distribution of cluster sizes, we consider the same directed polymer model described before by us [10]. The model is briefly outlined here for completeness. Consider a directed polymer on a discrete 'hyper-cubic' structure with random potential $\mu(x,t)$ assigned to each site (x,t) where x is the $(d-1)$-dimensional transverse vector and t is the longitudinal length of the polymer in the direction for which no reverse step is allowed. The walk is restricted by $|x(t) - x(t+1)| = 0$ or 1, but there is a biassing or bending factor γ against

walks such that $|x(t) - x(t+1)| = 1$. At zero temperature the minimum energy $E(x,t)$ of the polymer ending at (x,t) is calculated recursively, for example in $d = 1+1$

$$E(x,t) = \text{Min}[E(x,t-1) + \mu(x,t-1),\ E(x+1,t-1) + \mu(x+1,t-1) + \gamma,$$
$$E(x-1,t-1) + \mu(x-1,t-1) + \gamma] \tag{9}$$

where Min takes the minimum value. In the transverse direction the lattice has periodic boundary conditions. Using equation (9) the origin of the optimal path $x_0 = x_0(x,t)$ is recorded for each (x,t). In analogy with the growth model, a cluster is defined as a collection of all the points (x,t) whose origins are the same. Since $\langle (x - x_0)^2 \rangle^{1/2} \sim t^{1/z}$, the width of each cluster is expected to grow as $t^{1/z}$. The number of clusters of size s are measured at $t = 200$ on system size $L = 100\,000$ $(d = 1+1)$, $L = 500$ $(d = 2+1)$ and $L = 100$ $(d = 3+1)$ with twenty independent runs. The $\ln n(s)$ versus $\ln s$ curves are shown in figure 1. From the least squares fit to the data we get

$$\begin{aligned} \tau &= 1.40 \pm 0.02 \qquad d = 1+1 \\ \tau &= 1.55 \pm 0.02 \qquad d = 2+1 \\ \tau &= 1.63 \pm 0.02 \qquad d = 3+1. \end{aligned} \tag{10}$$

These values are in a perfect agreement with those given by equation (7): $\tau = 1.400$ $(d = 1+1)$, $\tau \approx 1.556$ $(d = 2+1)$ and $\tau \approx 1.643$ $(d = 3+1)$ with $z \approx 2(d+1)/(d+2)$ [9, 10]. Notice that these results are the same as those obtained from the Eden model and the ballistic deposition model in $d = 1+1$ and $d = 2+1$ confirming that both the growth models and the directed polymer in a random potential problem belong to the same universality class [7, 11]. There is an assumption that the individual clusters are compact for the derivation of equation (7), the above results support it even in $d = 3+1$.

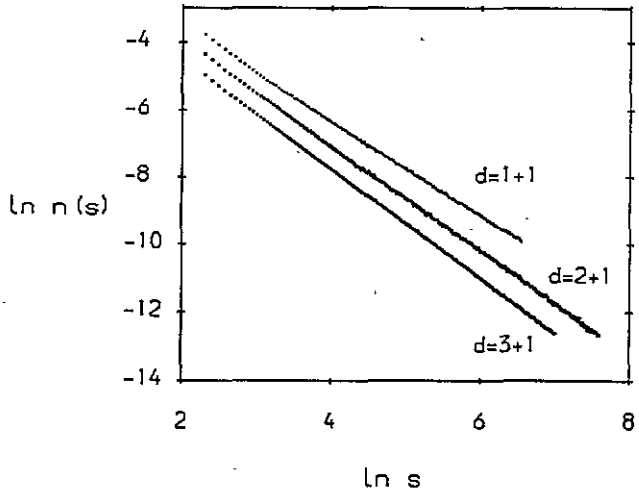

Figure 1. Cluster size distribution $n(s)$ as a function of size s on the directed polymer in a random potential. -0.5 $(d = 2+1)$ and -1 $(d = 3+1)$ are added to $\ln n(s)$ to avoid overcrowding.

Kondoh *et al* [12] studied Scheidegger's river network model [13] and found that $\nu_\perp = 1/3$ and $\nu_\parallel = 2/3$ which are different from the values of the Eden model. Since $z = 2$ of the Scheidegger's model are different from $z = 3/2$ of the Eden model in $d = 1 + 1$, they should belong to different universality classes.

There have been several studies to understand the drainage area of river networks [14]. The tree structure of each cluster would seem similar to that of a river network. In directed polymer language, the random potential $\mu(x, t)$ can be interpreted as the local height fluctuation from the tilted surface. Then the path of a river follows an optimal route minimizing the sum of the heights on the path to reduce the gravitational potential energy. Since the value $\nu_\parallel = 0.6$ of the growth models in $d = 1 + 1$ is in good agreement with Hack's measurement $2\nu_\parallel = 1.2 - 1.4$ on rivers [15], the directed polymer in a random potential can be a good model to describe the river basin.

In this letter, we have shown a relation between the critical exponents $\nu_\parallel$ and $\nu_\perp$ controlling the individual cluster and the dynamic critical exponent z of the surface structure. The numerical results of the exponents $\nu_\parallel$, $\nu_\perp$ and τ of the Eden model and τ for the directed polymer in a random potential are in excellent agreement with equation (7). It would be interesting to test the results of equation (7) on other models such as the restricted solid on solid growth model [9] in higher dimensions.

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